JOURNAL OF COMPUTATIONAL PHYSICS 114, 284–297 (1994)

Riemann Solver for Relativistic Hydrodynamics

DINSHAW S. BALSARA*

Physics and Astronomy Department, Johns Hopkins University, Baltimore, Maryland 21218

Received January 26, 1993; revised March 23, 1994

In this paper we construct an efficient, accurate, and rugged Riemann solver for relativistic hydrodynamics. The algorithm is an extension of the two shock approximation of Colella to the relativistic regime. The Riemann solver constructed here is made to converge to the solution via iteration. Two different iterative techniques are presented, one based on a secant method and the other on a Newton method. The method presented here provides an exact treatment of the transverse velocities across general, oblique shocks. This is a non-trivial but very desirable property to have in a Riemann solver for relativistic flow. We also show the equivalence of our new formulation to the previous ones in the non-relativistic limit. © 1994 Academic Press, Inc.

I. INTRODUCTION

In recent years relativistic hydrodynamics has come to play an important role both in science and technology. The applications in science mainly draw from astrophysics. Almost any high energy astrophysical phenomenon requires a relativistic treatment. Hence, applications in that field of study are legion and will not be catalogued here. Recent advances in technology have also shown the need for relativistic treatment in several non-astrophysical problems. Thus high energy particle beams, ultra-relativistic heavy ion collisions, high energy nuclear collisions, and free-electron laser technology all call for a relativistic hydrodynamic treatment. In all these problem areas there is some interest in computing out solutions rather than solving problems analytically. Attempts at a computational solution of relativistic hydrodynamics have so far relied on artificial viscosity formulations, Hawley, Smarr, and Wilson [8, 9], Norman and Winkler [12]. The net result of this effort led Norman and Winkler [12] to conclude that artificial viscosity-based formulations do not work on an eulerian grid. This automatically prevents us from making multi-dimensional schemes. Hawley, Smarr, and Wilson [9] go so far as to observe that the absence of a Riemann solver for relativistic hydrodynamics is a serious impediment to the computational study of relativistic fluid dynamics. We, therefore, begin a study of this important computational problem in this paper which is the first in a series of papers.

Reliable methods for computing non-relativistic compressible fluid dynamics have existed for the last decade. Schemes like that of Colella and Woodward [3] yield perfectly adequate results for a large variety of problems. It would not be unfair to say that most of these schemes are higher order Godunov-like schemes. The basic ingredients of such schemes inevitably consist of an interpolation step and a Riemann solver step. The interpolation can be imposed on the primitive variables, Colella and Woodward [3], on the characteristic variables, Harten *et al.* [7], or on the fluxes, Harten [5], Osher and Chakravarthy [13]. Various strategies for constructing the Riemann solver also exist. Thus one may construct an exact Riemann solver, van Leer [19], or an approximate Riemann solver. Approximate Riemann solvers usually consist of methods that are based on flux vector splitting or flux difference splitting. Experience has shown that the latter class of Riemann solvers yields answers that are superior to the former. Thus we focus on flux difference splitting strategies here.

One of the well-used flux difference strategies for solving the Riemann problem is the method of Roe [16]. It relies on making a local linearization of the difference in the fluxes on either side of a zone boundary and solving the corresponding linearized problem. Another is the method of Osher and Solomon [14] which relies on representing the Riemann problem with rarefaction waves and a contact discontinuity. Colella [2] has presented a third strategy which represents the simple waves in the Riemann problem by a pair of shocks and a contact discontinuity. Experience has shown this third strategy to be more rugged than the original exact Riemann solver of van Leer [19] and also capable of computing shocks of arbitrary strength, Colella [2], Woodward and Colella [20]. It is the most complicated of the three Riemann solvers catalogued in this paragraph. It is also the method of choice that production codes turn to when it is felt on reviewing a simulation's results that a simpler Riemann solver might have been the source of spurious entropy generation in the problem. We

* Currently at National Center for Supercomputing Applications, University of Illinois, Urbana-Champaign.

0021-9991/94 $6.00

also observe that the method of Colella [2] can be extended almost trivially to yield an exact Riemann solver. In a later section we will also show that it provides an exact treatment of situations where there is a transverse velocity across a shock. For non-relativistic hydrodynamics this is trivially achieved. For relativistic flow this is a non-trivial and very desirable property to have in a Riemann solver. For all these reasons we focus on extending the two-shock method to relativistic flow in this paper.

In this paper we devise a method of solving the Riemann problem for relativistic flow. Since any general relativistic metric can be reduced to a locally Lorentz metric at a zone boundary this method is useful for both special relativistic and general relativistic calculations. In Section II we catalogue some useful results from the theory of one-dimensional shocks. It is also an instructive way of introducing our notation for the rest of this paper since such notation varies widely from textbook to textbook. In Section III we build up some results that are useful for oblique shocks and show their usefulness in the Riemann problem. Almost all previous efforts at formulating the Riemann problem have been made using a lagrangian formulation. In this section we show that the problem can be formulated in an arbitrary lab-frame, too, which yields as it were an eulerian formulation. In Section IV we use those results to build two iterative root solvers for the Riemann problem for relativistic flow. In Section V we show that the iteration schemes converge to the exact solution at their designed rates for a few test case shock tube problems. These will later be used as test problems in Balsara [1]. Appendices A and B contain auxiliary derivations and results that are useful for this paper but are too long to include in the main body of the paper. Appendix C shows the equivalence of the present formulation in its non-relativistic limit to the lagrangian formulation of van Leer [19].

II. ONE-DIMENSIONAL STATIONARY SHOCKS

In this section we list without proof some relevant results from the theory of one dimensional relativistic shocks. All the results in this section hold in the shock's rest frame. We have found the paper by Taub [18] very useful for this part of the work. We have also found the books by Mihalas and Mihalas [11], Synge [17], and Landau and Lifshitz [10] very useful. Throughout this work our metric will be diag$\{-1, 1, 1, 1\}$. Any general relativistic metric can be locally transformed to this form. In the Landau and Lifshitz style subscripts 1 and 2 will always denote pre-shock and post-shock values, respectively. Many of the applications of relativistic flow can be represented quite well by a gamma-law gas. Hence, in this work we will focus on gamma-law gases. More general equations of state will require performing a sub-iteration on the equation of state as explained by Colella and Glaz [4]. In this work Γ denotes the polytropic index while γ is reversed for the Lorenz factor. In keeping with the style of van Leer [19], all equation numbers that are specific to a gamma-law gas will be appended with a Γ.

The one-dimensional conservative form of the equations of relativistic hydrodynamics can be written as

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho\gamma \\ \rho\mu\gamma^2 - P/c^2 \\ \rho\mu\gamma^2 v_x \\ \rho\mu\gamma^2 v_y \\ \rho\mu\gamma^2 v_z \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} \rho\gamma v_x \\ \rho\mu\gamma^2 v_x \\ \rho\mu\gamma^2 v_x^2 + P \\ \rho\mu\gamma^2 v_x v_y \\ \rho\mu\gamma^2 v_x v_z \end{pmatrix} = 0. \tag{2.1}$$

Here "c" denotes the speed of light. The speed of sound will be denoted by α; ρ, P, ε, e, and S are the density, pressure, mass-equivalent thermal energy, total energy density, and entropy, respectively, of the fluid in the fluid's rest frame. They are all world scalars. γ is the Lorentz factor defined in the usual way ($\gamma = 1/\sqrt{1-(v_x^2+v_y^2+v_z^2)/c^2}$); $\rho\mu$ gives the sum of the rest-frame density and the mass-equivalent enthalpy density where μ is defined by

$$\mu = 1 + \frac{\varepsilon}{c^2} + \frac{P}{\rho c^2} \tag{2.2}$$

$$\mu = 1 + \frac{\Gamma}{\Gamma - 1}\frac{P}{\rho c^2}. \tag{2.3Γ}$$

The equations can be substantially simplified if one sets

$$\mathbf{v}_{\text{new}} = \mathbf{v}_{\text{old}}/c, \qquad P_{\text{new}} = P_{\text{old}}/c^2, \qquad \varepsilon_{\text{new}} = \varepsilon_{\text{old}}/c^2, \tag{2.4}$$

and multiplies the time coordinate with c. We assume henceforth that this is done and we also ignore the subscript "new" all over. The speed of sound is given by, Landau and Lifshitz [10, Section 134],

$$\alpha = \left(\frac{dP}{de}\right)^{1/2}, \tag{2.5}$$

where the derivative is evaluated with the entropy held fixed. For a gamma-law gas we have

$$\alpha = \left(\frac{\Gamma P}{\rho\mu}\right)^{1/2}. \tag{2.6Γ}$$

Even in the relativistic case the equation for the pseudo-entropy $A(S)$ and mass-equivalent thermal energy of a gamma-law gas are given by

$$A(S) = \frac{P}{\rho^{\Gamma}}, \qquad \varepsilon = \frac{1}{\Gamma - 1}\frac{P}{\rho}. \tag{2.7Γ}$$

The total energy density is related to the density and mass-equivalent thermal energy by

$$e = \rho(1+\varepsilon). \tag{2.8}$$

The equations for mass, momentum, and energy conservation across a shock in the shock's rest frame are given by

$$\rho_1 \gamma_1 v_1 = \rho_2 \gamma_2 v_2 \tag{2.9}$$

$$\rho_1 \mu_1 \gamma_1^2 v_1^2 + P_1 = \rho_2 \mu_2 \gamma_2^2 v_2^2 + P_2 \tag{2.10}$$

$$\rho_1 \mu_1 \gamma_1^2 v_1 = \rho_2 \mu_2 \gamma_2^2 v_2. \tag{2.11}$$

For non-relativistic flow the Rankine–Hugoniot adiabat relates the post-shock thermodynamic variables to the pre-shock thermodynamic variables. The Taub adiabat performs a similar function in the relativistic case. It is given by, Taub [18],

$$(P_2 - P_1)\left(\frac{\mu_1}{\rho_1} + \frac{\mu_2}{\rho_2}\right) = \mu_2^2 - \mu_1^2. \tag{2.12}$$

Thus if we are given an equation of state and pre-shock density and pressure, then for a specified post-shock pressure Eq. (2.12) gives us a post-shock density. For a gamma-law gas we substitute ρ as a function of μ and P from Eq. (2.3Γ) to obtain

$$\frac{(\Gamma-1)}{\Gamma}(P_2 - P_1)\left(\frac{\mu_1(\mu_1 - 1)}{P_1} + \frac{\mu_2(\mu_2 - 1)}{P_2}\right) = \mu_2^2 - \mu_1^2. \tag{2.13Γ}$$

This makes the quadratic aspect of Eq. (2.13Γ) clear. Only one of the roots of the above quadratic is physical, as we show in Appendix A. The fluid's pre- and post-shock speeds in the shock's rest frame are given by, Landau and Lifshitz [10, Section 135],

$$v_1 = \sqrt{(P_2 - P_1)(e_2 + P_1)/(e_2 - e_1)(e_1 + P_2)} \tag{2.14}$$

$$v_2 = \sqrt{(P_2 - P_1)(e_1 + P_2)/(e_2 - e_1)(e_2 + P_1)}. \tag{2.15}$$

We see that with the pre-shock thermodynamic variables specified, only the post-shock pressure needs to be known, in order for us to be able to deduce the post-shock density and the fluid's pre- and post-shock speeds in the shock's rest frame. We will make considerable use of this fact when constructing the Riemann solver. Note, too, that v_1 and v_2 in Eqs. (2.14) and (2.15) are positive definite by convention. We adopt this convention here so that the pre- and post-shock fluid speeds in the shock's rest frame will always be positive definite. Whether the shock is a C_- shock or a C_+ shock is determined solely by whether the fluid in the shock's rest frame enters the shock from the left or from the right, respectively.

For purposes of implementation, things work out a bit better if Eq. (2.13Γ) is expressed in terms of the post-shock pressure and total energy density. This and the choice of the correct root of Eq. (2.13Γ) are treated in Appendix A. Also, partial and total derivatives of v_1 and v_2 with respect to P_2 become necessary in constructing the derivative terms in the iteration step of Newton's method iterative solver. These have been catalogued in Appendix B. To construct the total derivative of v_1 and v_2 with respect to P_2 we will also need the total derivative of e_2 with respect to P_2. That, too, has been done in Appendix A. The reader wishing to have a comprehensive understanding of this paper should read Appendices A and B at this point.

III. OBLIQUE SHOCKS

We begin by observing that in the non-relativistic case the shocks used in solving the Riemann problem at the zone boundaries of any Godunov-like scheme are almost always oblique shocks. The only exception is when the transverse velocities at the zone boundary are zero. We also note that for non-relativistic flow we can go from a specified stationary perpendicular shock to a range of oblique shocks with the same pressure jump simply by (1) starting in the rest-frame of the normal stationary shock with the specified pressure jump and pre-shock thermodynamic variables, (2) making a Galilean transformation in the plane of the shock so that the shock is still stationary in the new frame but has a transverse fluid velocity across it, and (3) making another Galilean transformation perpendicular to the plane of the shock. In the rest of this paper we assume that the zone boundary has its normal aligned along the x-axis. To make an oblique shock (say, for example, a C_- shock) with a specified pressure jump matchup with the lab-frame fluid velocity in front of the shock (i.e., to the left of the shock in our example), we start off in the rest frame of a shock with the specified pressure jump and pre-shock thermodynamic variables as in step (1) above. Then we make a Galilean transformation as in step (2) above with the velocity vector given by the negative of the transverse velocity in front of the shock. Last, we make a Galilean transformation as in step (3) above so that the pre-shock x-velocity matches up with the pre-shock x-velocity of the fluid ahead of the shock. The lab-frame can be identified with the zone boundary which is assumed to be non-moving. (In our example, if the fluid to the left of the shock moves with a lab-frame x-velocity given by v_{x1l} and if the pre-shock fluid speed in the shock's rest frame is given by v_{1L} then the Galilean transformation in step (3) above should be made with a velocity $-\eta$ such that $v_{1L} + \eta = v_{x1l}$. If the post-shock fluid speed in the shock's rest frame is given by v_{2L} and if the post-shock velocity in the lab-frame is denoted by v_{x2l} then $v_{x2l} = v_{2L} + (v_{x1l} - v_{1L})$.) The thing that is of real interest in this exercise is the post-shock x-velocity. Since appropriate velocities for the Galilean transformations in steps (2) and (3) above are easily determined, this quantity can be found rather easily. It is only when the post-shock x-velocity in the lab-frame for the left-going C_- shock matches up with that of the right-going C_+ shock (to some reasonable approximation) that we say that the Riemann solver has converged.

For relativistic flow we go through exactly the same steps with a slight twist. The twist comes about because the transverse velocity gets changed by $O(v^2/c^2)$ in a boosted frame. Thus we have to be sensitive to such changes. This is not a big problem when the Riemann solver is used for proper upwinding in regions of slow spatial variation, such as in a rarefaction fan. On the other hand, for strong isolated shocks the $O(v^2/c^2)$ contributions to the transverse velocity across the shock can become significant, i.e., of order unity. An important feature of the method presented here is that it treats transverse velocities across shocks of arbitrary strength exactly. In the rest of this section we build up formulae for C_- and C_+ shocks with arbitrary transverse velocities in relativistic flow.

C_- Shock

The C_- shock is a left-going shock. The fluid in the rest-frame of the shock flows into it from the left. We denote the pre- and post-shock fluid speeds in the rest-frame of the shock by v_{1L} and v_{2L}, respectively. The fluid variables in the lab-frame to the left of the left-going shock have a state vector given by $(\rho_{1l}, P_{1l}, v_{x1l}, v_{y1l}, v_{z1l})^{\mathrm{T}}$. We denote the rest-frame of the shock by S_L with coordinates (t, x, y, z). The lab-frame is denoted by S'' with coordinates (t'', x'', y'', z''). The pre-shock fluid four-velocity in S_L is given by

$$(\gamma_{1L}, \gamma_{1L} v_{1L}, 0, 0)^{\mathrm{T}} \qquad (\text{where } \gamma_{1L} \equiv 1/\sqrt{1 - v_{1L}^2}) \tag{3.1}$$

and the pre-shock fluid four-velocity in S'' is given by

$$(\gamma_{1l}, \gamma_{1l} v_{x1l}, \gamma_{1l} v_{y1l}, \gamma_{1l} v_{z1l})^{\mathrm{T}} \qquad (\text{where } \gamma_{1l} \equiv 1/\sqrt{1 - v_{x1l}^2 - v_{y1l}^2 - v_{z1l}^2}). \tag{3.2}$$

Consider a frame S'_L moving with speed ξ_l in a direction given by

$$-n_{yl}\hat{y} - n_{zl}\hat{z} \qquad (\text{where } n_{yl} \equiv v_{y1l}/\sqrt{v_{y1l}^2 + v_{z1l}^2},\; n_{zl} \equiv v_{z1l}/\sqrt{v_{y1l}^2 + v_{z1l}^2}), \tag{3.3}$$

with respect to S_L. Let S'_L have coordinates (t', x', y', z'). The four-velocity of the fluid in S'_L is given by

$$(\gamma_{\xi l}\gamma_{1L}, \gamma_{1L} v_{1L}, \gamma_{\xi l}\gamma_{1L}\xi_l n_{yl}, \gamma_{\xi l}\gamma_{1L}\xi_l n_{zl})^{\mathrm{T}}. \tag{3.4}$$

Consider yet another frame S''_L moving with velocity $-\eta_l \hat{x}'$ in S'_L. The fluid four-velocity in S''_L is given by

$$\begin{pmatrix} \gamma_{\eta l}\gamma_{\xi l}\gamma_{1L} + \gamma_{\eta l}\gamma_{1L}\eta_l v_{1L} \\ \gamma_{\eta l}\gamma_{\xi l}\gamma_{1L}\eta_l + \gamma_{\eta l}\gamma_{1L} v_{1L} \\ \gamma_{\xi l}\gamma_{1L}\xi_l n_{yl} \\ \gamma_{\xi l}\gamma_{1L}\xi_l n_{zl} \end{pmatrix}. \tag{3.5}$$

The frame S''_L can be identified with the lab-frame S'' if the four velocities in Eqs. (3.2) and (3.5) are equal. This is what determines the magnitudes of the two boosts that we have made above. Thus we obtain

$$\gamma_{1l} = \gamma_{\eta l}\gamma_{\xi l}\gamma_{1L} + \gamma_{\eta l}\gamma_{1L}\eta_l v_{1L} \tag{3.6}$$

$$\gamma_{1l} v_{x1l} = \gamma_{\eta l}\gamma_{\xi l}\gamma_{1L}\eta_l + \gamma_{\eta l}\gamma_{1L} v_{1L} \tag{3.7}$$

$$\gamma_{1l} v_{y1l} = \gamma_{\xi l}\gamma_{1L}\xi_l n_{yl} \tag{3.8}$$

$$\gamma_{1l} v_{z1l} = \gamma_{\xi l}\gamma_{1L}\xi_l n_{zl}. \tag{3.9}$$

Squaring and adding Eqs. (3.8) and (3.9) gives us

$$\gamma_{\xi l}^2 \equiv \frac{1}{1 - \xi_l^2} = \frac{\gamma_{1l}^2}{\gamma_{1L}^2}(v_{y1l}^2 + v_{z1l}^2) + 1. \tag{3.10}$$

Taking the ratio of Eqs. (3.6) and (3.7) we obtain

$$\eta_l = \frac{v_{x1l} - (v_{1L}/\gamma_{\xi l})}{1 - v_{x1l}(v_{1L}/\gamma_{\xi l})}. \tag{3.11}$$

Thus both the transformations have been pinned down and the frame S''_L can be identified with the lab-frame S''.

We denote the fluid's post-shock state vector in the lab-frame S'' by $(\rho_{2l}, P_{2l}, v_{x2l}, v_{y2l}, v_{z2l})^{\mathrm{T}}$. For a given P_{2l} we can get the density ρ_{2l} from the Taub adiabat. The fluid's post-shock four velocity in the shock's rest-frame S_L is given by

$$(\gamma_{2L}, \gamma_{2L} v_{2L}, 0, 0)^{\mathrm{T}} \qquad (\text{where } \gamma_{2L} \equiv 1/\sqrt{1 - v_{2L}^2}). \tag{3.12}$$

Thus by making the same set of Lorentz transformations from S_L to S'_L to S''_L (which is just S'') the fluid's post-shock velocity in the lab-frame is given by

$$v_{x2l} = \frac{\eta_l + (v_{2L}/\gamma_{\xi l})}{1 + \eta_l(v_{2L}/\gamma_{\xi l})} \tag{3.13}$$

$$v_{y2l} = v_{y1l}\frac{(\gamma_{\xi l} + \eta_l v_{1L})}{(\gamma_{\xi l} + \eta_l v_{2L})} \tag{3.14}$$

$$v_{z2l} = v_{z1l}\frac{(\gamma_{\xi l} + \eta_l v_{1L})}{(\gamma_{\xi l} + \eta_l v_{2L})}. \tag{3.15}$$

Thus we see that for strongly relativistic shocks by v_{y2l} and

v_{y1l} can differ substantially. One way of feeling comfortable with this result is to realize that the transverse velocity is not preserved under Lorentz transformations the way that it is for Galilean transformations.

Another way of feeling comfortable with this result is to realize that the number density and transverse momentum density have different factors of γ and μ in them; see Eq. (2.1). Now the ratio of the transverse momentum density to the mass density carries an extra term $\mu_{2l}\gamma_{2l}$ behind the shock which is different from $\mu_{1l}\gamma_{1l}$ in front of the shock and so the transverse velocities behind a (compression) shock drop just to ensure that the specific transverse momentum is conserved.

The shock speed in the lab-frame is also easily deduced. Realize that the locus of the shock in the frame S_L is given by $x=0$. A sequence of Lorentz transforms shows us that the shock position in the frame S'' is given by

$$x'' = \eta_l t''. \tag{3.16}$$

C_+ Shock

The C_+ shock is a right-going shock. The fluid in the rest-frame of the shock flows into it from the right. We denote the pre- and post-shock fluid speeds in the rest-frame of the shock by v_{1R} and v_{2R}, respectively. The fluid variables in the lab-frame to the right of the right-going shock have a state vector given by $(\rho_{1r}, P_{1r}, v_{x1r}, v_{y1r}, v_{z1r})^{\mathrm{T}}$. We denote the rest-frame of the shock by S_R with coordinates (t, x, y, z). The lab-frame is denoted by S'' with coordinates (t'', x'', y'', z''). The pre-shock fluid four-velocity in S_R is given by

$$(\gamma_{1R}, -\gamma_{1R} v_{1R}, 0, 0)^{\mathrm{T}} \qquad (\text{where } \gamma_{1R} \equiv 1/\sqrt{1-v_{1R}^2}) \tag{3.17}$$

and the pre-shock fluid four-velocity in S'' is given by

$$(\gamma_{1r}, \gamma_{1r} v_{x1r}, \gamma_{1r} v_{y1r}, \gamma_{1r} v_{z1r})^{\mathrm{T}}$$
$$(\text{where } \gamma_{1r} \equiv 1/\sqrt{1-v_{x1r}^2 - v_{y1r}^2 - v_{z1r}^2}). \tag{3.18}$$

Consider a frame S'_R moving with speed ξ_r in a direction given by

$$-n_{yr}\hat{y} - n_{zr}\hat{z} \qquad (\text{where } n_{yr} \equiv v_{y1r}/\sqrt{v_{y1r}^2 + v_{z1r}^2},$$
$$n_{zr} \equiv v_{z1r}/\sqrt{v_{y1r}^2 + v_{z1r}^2}) \tag{3.19}$$

with respect to S_R. Let S'_R have coordinates (t', x', y', z'). The four-velocity of the fluid in S'_R is given by

$$(\gamma_{\xi r}\gamma_{1R}, -\gamma_{1R} v_{1R}, \gamma_{\xi r}\gamma_{1R}\xi_r n_{yr}, \gamma_{\xi r}\gamma_{1R}\xi_r n_{zr})^{\mathrm{T}}. \tag{3.20}$$

Consider yet another frame S''_R moving with velocity $-\eta_r\hat{x}'$ in S'_R. The fluid four-velocity in S''_R is given by

$$\begin{pmatrix} \gamma_{\eta r}\gamma_{\xi r}\gamma_{1R} - \gamma_{\eta r}\gamma_{1R}\eta_r v_{1R} \\ \gamma_{\eta r}\gamma_{\xi r}\gamma_{1R}\eta_r - \gamma_{\eta r}\gamma_{1R} v_{1R} \\ \gamma_{\xi r}\gamma_{1R}\xi_r n_{yr} \\ \gamma_{\xi r}\gamma_{1R}\xi_r n_{zr} \end{pmatrix}. \tag{3.21}$$

As in the case of the C_- shock the frame S''_R can be identified with the lab-frame S'' if the four velocities in Eqs. (3.18) and (3.21) are equal. Just as before, this requires us to make the following choices for magnitudes of the two boosts that we have made:

$$\gamma_{\xi r}^2 \equiv \frac{1}{1-\xi_r^2} = \frac{\gamma_{1r}^2}{\gamma_{1R}^2}(v_{y1r}^2 + v_{z1r}^2) + 1 \tag{3.22}$$

and

$$\eta_r = \frac{v_{x1r} + (v_{1R}/\gamma_{\xi r})}{1 + v_{x1r}(v_{1R}/\gamma_{\xi r})}. \tag{3.23}$$

Thus both the transformations have been pinned down and the frame S''_R can be identified with the lab-frame S''.

We denote the fluid's post-shock state vector in the lab-frame S'' by $(\rho_{2r}, P_{2r}, v_{x2r}, v_{y2r}, v_{z2r})^{\mathrm{T}}$. As before, for a given P_{2r} we can obtain the density ρ_{2r} from the Taub adiabat. The fluid's post-shock four-velocity in the shock's rest-frame S_R is given by

$$(\gamma_{2R}, -\gamma_{2R} v_{2R}, 0, 0)^{\mathrm{T}} \qquad (\text{where } \gamma_{2R} \equiv 1/\sqrt{1-v_{2R}^2}). \tag{3.24}$$

Thus by making the same set of Lorentz transformations from S_R to S'_R to S''_R (which is just S'') the fluid's post-shock velocity in the lab-frame is given by

$$v_{x2r} = \frac{\eta_r - (v_{2R}/\gamma_{\xi r})}{1 - \eta_r(v_{2R}/\gamma_{\xi r})} \tag{3.25}$$

$$v_{y2r} = v_{y1r}\frac{(\gamma_{\xi r} - \eta_r v_{1R})}{(\gamma_{\xi r} - \eta_r v_{2R})} \tag{3.26}$$

$$v_{z2r} = v_{z1r}\frac{(\gamma_{\xi r} - \eta_r v_{1R})}{(\gamma_{\xi r} - \eta_r v_{2R})}. \tag{3.27}$$

Just as in the previous case, the locus of the shock in the frame S_R is given by $x=0$. A sequence of Lorentz transforms shows us that the shock position in the frame S'' is given by

$$x'' = \eta_r t''. \tag{3.28}$$

IV. CONSTRUCTING THE ROOT SOLVER

Solving the Riemann problem for non-relativistic hydrodynamics essentially consists of fitting a left-going simple wave and a right-going simple wave to the specified left and right states so that the normal velocities on either side of the contact discontinuity match up. Collela [2] has shown that a rugged method can be devised by restricting those simple waves to be shocks. Thus instead of using a rarefaction wave (when it is necessary) one uses the analytical continuation of a shock wave Hugoniot onto the rarefaction side. Proper entropy enforcement at rarefaction shocks can then be obtained via an entropy fix as shown by Harten and Hyman [6]. All of these facts carry over to the relativistic case. The essential part of solving the Riemann problem is, therefore, to guess a post-shock pressure P_2 such that the post-shock x-velocities on either side of the contact discontinuity are equal. This requirement translates into finding the root of

$$v_{x2l} - v_{x2r} = 0. \tag{4.1}$$

In general, if the equation is rather complicated this task is entrusted to a root solver algorithm. In the next two subsections we construct two root solvers, one based on a secant method and the other based on a Newton method. The secant method is simple and easy to implement and has a good rate of convergence. The Newton method is harder to implement but has the same convergence rate as the highly efficient non-relativistic Riemann solvers of van Leer [19] and Colella [2].

(1) *The Secant Method*

The secant method, as defined in van Leer [19 and references therein], consists of fitting two straight lines from the points (P_2^n, v_{x2l}^n) and (P_2^n, v_{x2r}^n) to the points (P_{1l}, v_{x1l}) and (P_{1r}, v_{x1r}) on the respective Hugoniot curves. The point of intersection of the two lines gives us the next iterate P_2^{n+1}. This is a somewhat different recipe from that provided in Press *et al.* [15]. It is, however, the way the secant method is routinely used in numerical hydrodynamics. The nth iteration of the secant method proceeds via the following steps:

Step I. Let P_2^n denote the pressure obtained from the previous iteration. Using P_2^n in Eq. (A.9Γ) find the energy densities and, therefore, the other required thermodynamic variables behind the shocks on either side. Use Eqs. (2.14) and (2.15) to obtain v_{1L}, v_{2L}, v_{1R}, and v_{2R}.

Step II. Using Eqs. (3.11) and (3.13) obtain v_{x2l}. Similarly, using Eqs. (3.23) and (3.25) obtain v_{x2r}. These are the values for the post-shock x-velocities in the lab-frame for that iteration.

Step III. Obtain the new iterate P_2^{n+1} from

$$P_2^{n+1} = \frac{[v_{x1r} - v_{x1l}] + \dfrac{dv_{x2l}^n}{dP_2^n} P_{1l} - \dfrac{dv_{x2r}^n}{dP_2^n} P_{1r}}{\left[\dfrac{dv_{x2l}^n}{dP_2^n} - \dfrac{dv_{x2r}^n}{dP_2^n}\right]}, \tag{4.2}$$

where

$$\frac{dv_{x2l}^n}{dP_2^n} = \frac{v_{x2l}^n - v_{x1l}}{P_2^n - P_{1l}}, \qquad \frac{dv_{x2r}^n}{dP_2^n} = \frac{v_{x2r}^n - v_{x1r}}{P_2^n - P_{1r}}. \tag{4.3}$$

The iteration can be started using

$$\frac{dv_{x2l}^0}{dP_2^0} = \frac{v_{x2l}^0 - v_{x1l}}{P_{2l}^0 - P_{1l}}, \qquad \frac{dv_{x2r}^0}{dP_2^0} = \frac{v_{x2r}^0 - v_{x1r}}{P_{2r}^0 - P_{1r}} \tag{4.4}$$

in Eq. (4.2). Here $P_{2l}^0 = (1+\delta)\, P_{1l}$, $P_{2r}^0 = (1+\delta)\, P_{1r}$, and δ is a small number, say, 0.05; v_{x2l}^0 and v_{x2r}^0 are the post-shock velocities corresponding to P_{2l}^0 and P_{2r}^0.

(2) *The Newton Method*

The Newton method for obtaining the roots of a transcendental equation has been shown to have second-order convergence. It is, therefore, the method of choice. Van Leer [19] has shown that a root solver based on the Newton method can be constructed for non-relativistic flow. We show here that the same can be done for relativistic flow. The Newton method consists of evaluating the total derivative of the left-hand side of Eq. (4.1) with respect to the independent variable, which in our case is the pressure at the contact discontinuity P_2, and of using that to iterate P_2 to convergence. Thus, using a superscript "n" for the nth iterate, we have the following iteration scheme:

$$P_2^{n+1} = P_2^n - [v_{x2l}^n - v_{x2r}^n] \Big/ \left[\frac{dv_{x2l}^n}{dP_2^n} - \frac{dv_{x2r}^n}{dP_2^n}\right]. \tag{4.5}$$

In order to implement the iteration scheme we need analytical forms for the total derivatives in the denominator of Eq. (4.5). Observing Eqs. (3.13) and (3.25) we note that they are functions of $(\eta_l, \gamma_{\xi l}, v_{2L})$ and $(\eta_r, \gamma_{\xi r}, v_{2R})$, respectively. Equations (3.10) and (3.11) show us that η_l and $\gamma_{\xi l}$ are functions of v_{1L}. Similarly, Eqs. (3.22) and (3.23) show us that η_r and $\gamma_{\xi r}$ are functions of v_{1R}. The total derivatives of v_{1L}, v_{2L}, v_{1R}, and v_{2R} with respect to P_2 have been constructed in Appendices A and B. Thus the problem of obtaining an analytical representation of the total derivatives in Eq. (4.5) can be reduced to successive application of the chain rule.

For the left-going shock we have

$$\frac{d\gamma_{\xi l}}{dv_{1L}} = -\frac{v_{1L}}{\gamma_{\xi l}} \gamma_{1l}^2 (v_{y1l}^2 + v_{z1l}^2). \tag{4.6}$$

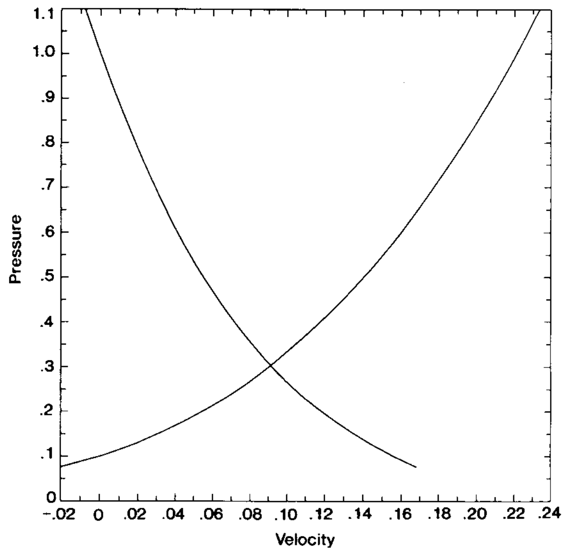

FIG. 1. Hugoniots for the modified Sod shock with $\rho_{1l}=100$, $P_{1l}=1$ and $\rho_{1r}=12.5$, $P_{1r}=0.1$. Two shock approximation used.

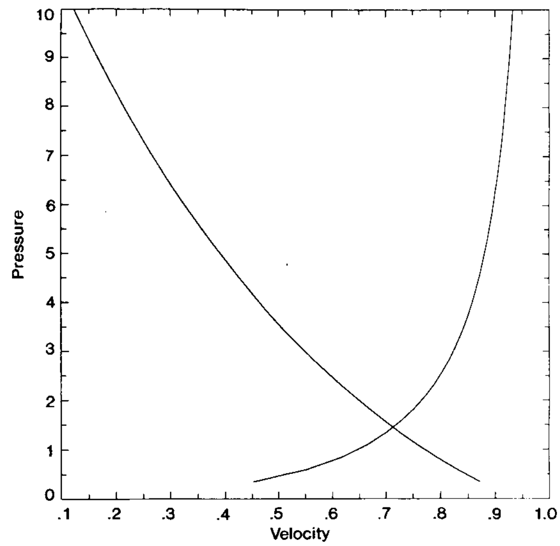

FIG. 2. Hugoniots for the relativistic shock tube, Hawley, Smarr, and Wilson [8]. Here $\rho_{1l}=10$, $P_{1l}=40/3$ and $\rho_{1r}=1$, $P_{1r}=2\times10^{-6}/3$. Two shock approximation used.

here. We also wish to have a ruggedized relativistic Riemann solver for use in Balsara [1]. For all these test problems the speed of light is assumed to be unity.

The first test problem is a slight modification of the Sod shock problem. It consists of taking $\rho_{1l}=100$, $P_{1l}=1$ and $\rho_{1r}=12.5$, $P_{1r}=0.1$. The velocities on either side are initially set to zero. The polytropic index Γ is set to 1.4. The cell-break problem results in a non-relativistic flow. The solution for this problem is well known. The Hugoniot curves for the two shock approximation are shown in Fig. 1. Table I gives the convergence history for the Newton method applied to this problem. The first column gives the iteration number. The second and third columns give the iterated values of the pressure and velocity of the contact discontinuity. The fourth and fifth columns give the densities on the left and right of the contact discontinuity evaluated in the two shock approximation. The sixth column gives the order of convergence evaluated from successive iterates. The Riemann solver was asked to convergence to six significant digits of accuracy and the results were compared to a solution obtained with 10 digits of accuracy. We see that the Newton method converges at the design rate. The numbers that it converges to are also consistent with those expected for the standard Sod shock tube with the densities scaled up by a factor of 100 and the velocity scaled down by a factor of 10. Thus we retrieve the expected result in the non-relativistic limit.

The second test problem is the relativistic shock tube problem described in Hawley, Smarr, and Wilson [8]. It consists of taking $\rho_{1l}=10$, $P_{1l}=40/3$ and $\rho_{1r}=1$, $P_{1r}=2\times10^{-6}/3$. The velocities on either side are initially set to zero. The polytropic index Γ is set to $\frac{5}{3}$. The cell-break problem results in a relativistic flow. Owing to the great pressure difference on either side of the cell break problem one expects that several iterations will be needed in order for convergence to be obtained. The Hugoniot curves for the two shock approximation are shown in Fig. 2. Tables II and III give the convergence histories for the Newton and Secant methods applied to this problem. The columns in these tables are annotated similarly to Table I. Again the

TABLE I

Newton Method, Non-Relativistic Shock Tube

iterate	Pressure	Velocity	Density Left	Density Right	Convergence
1	1.9022853E-01	0.0000000E+00	1.0000000E+02	1.2500000E+01	
2	2.8451664E-01	1.2158817E-01	3.4527238E+01	1.9638602E+01	
3	3.0322208E-01	9.5597512E-02	4.3038328E+01	2.5552409E+01	2.0804
4	3.0369685E-01	9.1233176E-02	4.4695010E+01	2.6580572E+01	2.0067
5	3.0369713E-01	9.1125150E-02	4.4736926E+01	2.6606123E+01	2.0056
6	3.0369713E-01	9.1125150E-02	4.4736926E+01	2.6606123E+01	

TABLE II

Newton Method, Relativistic HSW Shock Tube

iterate	Pressure	Velocity	Density Left	Density Right	Convergence
1	4.5356245E-04	0.0000000E+00	1.0000000E+01	1.0000000E+00	
2	4.5287618E-02	9.4269063E-01	1.1123768E+00	3.9785000E+00	
3	3.9174192E-01	9.3157996E-01	1.2272288E+00	4.0417886E+00	9.0250
4	1.0047695E+00	8.6332104E-01	1.8349091E+00	4.3385640E+00	3.0776
5	1.3811611E+00	7.6965264E-01	2.5643930E+00	4.7870758E+00	2.2729
6	1.4392987E+00	7.2043461E-01	2.9352550E+00	5.0297306E+00	2.0549
7	1.4402745E+00	7.1323751E-01	2.9894436E+00	5.0654989E+00	2.0038
8	1.4402748E+00	7.1311754E-01	2.9903471E+00	5.0660958E+00	1.9853
9	1.4402748E+00	7.1311754E-01	2.9903471E+00	5.0660958E+00	

Riemann solver was asked to converge to six significant digits of accuracy and the results were compared to a solution obtained with 10 digits of accuracy. Hawley, Smarr, and Wilson [8] have obtained an analytic solution for this problem with the rarefaction fan exactly represented. They find a resolved pressure of 1.38 as opposed to our 1.44. This really shows that the two shock approximation works quite well, indeed, even when the problem is relativistic. Note from Table II that the first few iterations of the Newton method converge faster than the design rate of 2. The asymptotic order of convergence of the method remains 2 for all the iterations. From Table III we can see that the first few iterations of the secant method also converge rapidly. If the method were a true secant method it should have converged at a rate of 1.618. However, the comment made at the beginning of sub-section 4.1 shows that the secant method as used in numerical hydrodynamics is not a real secant method. Thus as the iterations progress, the rate of convergence soon deteriorates from the design rate for a real secant method. This should not be viewed very negatively

TABLE III

Secant Method, Relativistic Shock Tube

iterate	Pressure	Velocity	Density Left	Density Right	Convergence
1	4.5325209E-04	-8.5933702E-07	1.0000012E+01	1.0000012E+00	
2	2.3144068E-02	9.4269071E-01	1.1123759E+00	3.9784848E+00	
3	1.6433720E-01	9.3693314E-01	1.1728049E+00	4.0211427E+00	6.6071
4	4.4127010E-01	9.0566509E-01	1.4728236E+00	4.1484820E+00	2.3313
5	7.4359914E-01	8.5485672E-01	1.9037058E+00	4.3779452E+00	1.4731
6	9.9164554E-01	8.0702178E-01	2.2796470E+00	4.6054541E+00	1.2211
7	1.1659441E+00	7.7146033E-01	2.5507165E+00	4.7782326E+00	1.1176
8	1.2778312E+00	7.4798384E-01	2.7278994E+00	4.8934754E+00	1.0653
9	1.3459222E+00	7.3347686E-01	2.8371047E+00	4.9650934E+00	1.0368
10	1.3860878E+00	7.2484568E-01	2.9020555E+00	5.0078432E+00	1.0208
11	1.4093576E+00	7.1982086E-01	2.9398749E+00	5.0327780E+00	1.0118
12	1.4227004E+00	7.1693171E-01	2.9616252E+00	5.0471305E+00	1.0067
13	1.4303061E+00	7.1528225E-01	2.9740451E+00	5.0553298E+00	1.0038
14	1.4346271E+00	7.1434431E-01	2.9811081E+00	5.0599937E+00	1.0021
15	1.4370774E+00	7.1381219E-01	2.9851155E+00	5.0626403E+00	1.0012
16	1.4384653E+00	7.1351069E-01	2.9873861E+00	5.0641400E+00	1.0007
17	1.4392509E+00	7.1333999E-01	2.9886718E+00	5.0649891E+00	1.0004
18	1.4396956E+00	7.1324338E-01	2.9893994E+00	5.0654697E+00	1.0002
19	1.4399471E+00	7.1318871E-01	2.9898111E+00	5.0657417E+00	1.0001
20	1.4400894E+00	7.1315779E-01	2.9900440E+00	5.0658955E+00	1.0001
21	1.4401699E+00	7.1314029E-01	2.9901758E+00	5.0659826E+00	1.0000
22	1.4402155E+00	7.1313040E-01	2.9902503E+00	5.0660318E+00	1.0000
23	1.4402412E+00	7.1312480E-01	2.9902925E+00	5.0660596E+00	1.0000
24	1.4402558E+00	7.1312163E-01	2.9903163E+00	5.0660754E+00	1.0000
25	1.4402640E+00	7.1311984E-01	2.9903298E+00	5.0660843E+00	1.0000
26	1.4402687E+00	7.1311883E-01	2.9903375E+00	5.0660894E+00	1.0000
27	1.4402713E+00	7.1311826E-01	2.9903418E+00	5.0660922E+00	1.0000
28	1.4402728E+00	7.1311793E-01	2.9903442E+00	5.0660938E+00	1.0000
29	1.4402737E+00	7.1311775E-01	2.9903456E+00	5.0660947E+00	1.0000
30	1.4402737E+00	7.1311775E-01	2.9903456E+00	5.0660947E+00	

TABLE IV

Newton Method, First Relativistic Noh Shock Tube

iterate	Pressure	Velocity	Density Left	Density Right	Convergence
1	3.1628490E-03	8.9481927E-01	4.4642857E-01	4.4642857E-01	
2	3.4062203E-01	8.7932848E-01	1.7872579E+00	1.7872579E+00	
3	1.5943632E+00	6.3839033E-01	2.0622095E+00	2.0622095E+00	5.7838
4	2.4342799E+00	2.0537132E-01	2.7546544E+00	2.7546544E+00	2.1388
5	2.6142628E+00	3.1578275E-02	3.1002071E+00	3.1002071E+00	2.0119
6	2.6202445E+00	9.8508885E-04	3.1674348E+00	3.1674348E+00	2.0009
7	2.6202507E+00	1.0169624E-06	3.1696350E+00	3.1696350E+00	1.9921
8	2.6202507E+00	1.1130282E-12	3.1696372E+00	3.1696372E+00	

for the following three reasons: (1) The secant method is much easier to program. (2) The initial few digits of accuracy are obtained at a very rapid rate of convergence. In a numerical code one rarely requires more than a few digits of accuracy. (3) The construction of analytically defined derivatives for the Newton method in a numerical code takes several float point operations (unlike the non-relativistic case). Both the secant and Newton methods converge unconditionally for convex equations of state. However, if rapid convergence to arbitrary accuracy is desired, the Newton method is the method of choice.

The third and fourth test problems are derived from the non-relativistic Noh problem. A simplified version of the non-relativistic Noh problem may be derived by having two equal density pressureless fluids slam into each other with equal and opposite velocities. Our third and fourth test problems consist of having two fluid streams with Lorenz factors of 2.24 and 1000, respectively, run into each other with equal and opposite velocities. The Lorenz factor of 2.24 in our third test problem is barely within the range of Lorenz factors that can be treated by time-explicit, artificial viscosity-based formulations of relativistic hydrodynamics, Hawley, Smarr, and Wilson [8]. The Lorenz factor of 1000 in our fourth test problem is well outside the range of Lorenz factors that can be handled by time-explicit artificial viscosity formulations. The parameters for the fourth test problem, i.e., the one with a Lorenz factor of 1000, are chosen to be extremely concordant with those that would be expected in a gamma ray burst problem. The third test problem can be specified in detail by having $\rho_{1l}=1/(2.24)$, $P_{1l}=2\times10^{-6}/3$, $v_{1l}=(1-1/(2.24)^2)^{1/2}$ and $\rho_{1r}=1/(2.24)$, $P_{1r}=2\times10^{-6}/3$, $v_{1r}=-(1-1/(2.24)^2)^{1/2}$ and setting Γ the polytropic index to $\frac{5}{3}$. The fourth test problem can be specified in detail by having $\rho_{1l}=1/(1000)$, $P_{1l}=2\times10^{-6}/3$, $v_{1l}=(1-1/(1000)^2)^{1/2}$ and $\rho_{1r}=1/(1000)$, $P_{1r}=2\times10^{-6}/3$, $v_{1r}=-(1-1/(1000)^2)^{1/2}$ and setting Γ the polytropic index to $\frac{4}{3}$. The convergence histories for the Newton method applied to the third and fourth test problems are given in Tables IV and V, respectively. One readily sees that the convergence is rapid and occurs at the optimal rate. Convergence to the first two or three digits of accuracy is in fact extremely rapid.

The fifth test problem is taken from the stated values in Norman and Winkler [12]. It is specified by $\rho_{1l}=1$, $P_{1l}=1000$ and $\rho_{1r}=1$, $P_{1r}=2\times10^{-6}/3$. The velocities on either side are initially set to zero. The polytropic index Γ is set to $\frac{4}{3}$. The convergence tests are shown in Table VI. M. Norman has informed the author that the actual simulations presented in Norman and Winkler [12] were done with a polytropic index of $\frac{5}{3}$. We have run the Riemann

TABLE V

Newton Method, Second Relativistic Noh Shock Tube

iterate	Pressure	Velocity	Density Left	Density Right	Convergence

TABLE VI

Newton Method, Relativistic NW Shock Tube

iterate	Pressure	Velocity	Density Left	Density Right	Convergence
1	6.0980714E-02	0.0000000E+00	1.0000000E+00	1.0000000E+00	
2	4.6247401E-01	9.9983439E-01	4.5494407E-03	3.3495450E+00	
3	1.5989877E+00	9.9875969E-01	1.2461910E-02	6.5912623E+00	2.9830
4	3.9600407E+00	9.9573464E-01	2.3171227E-02	8.5369504E+00	2.3697
5	8.1060808E+00	9.8949969E-01	3.6555196E-02	1.0786424E+01	2.3948
6	1.2972233E+01	9.7869560E-01	5.2566951E-02	1.3532264E+01	2.4262
7	1.5238135E+01	9.6623873E-01	6.6906396E-02	1.6002798E+01	2.2504
8	1.5452391E+01	9.6051807E-01	7.2720921E-02	1.6999945E+01	2.0530
9	1.5453823E+01	9.5997971E-01	7.3250011E-02	1.7090459E+01	2.0024
10	1.5453823E+01	9.5997612E-01	7.3253535E-02	1.7091061E+01	1.1374
11	1.5453823E+01	9.5997612E-01	7.3253535E-02	1.7091061E+01	

solver with the above values for the pressure and density but a polytropic index of $\frac{5}{3}$ and found it to agree very well with their results.

APPENDIX A: SOME USEFUL EQUATIONS DERIVED FROM THE TAUB ADIABAT

In this section we do two things: (1) We build up a formula for e_2 given e_1, P_1, and P_2. (2) We build up a formula for the total derivative of e_2 with respect to P_2. From Eqs. (2.2) and (2.8) we obtain

$$\rho\mu = e + P. \tag{A.1}$$

And for a gamma-law gas we have

$$\rho = \frac{(\Gamma-1)\,e - P}{(\Gamma-1)} \tag{A.2Γ}$$

$$\mu = \frac{(\Gamma-1)(e+P)}{(\Gamma-1)\,e-P}, \tag{A.3Γ}$$

so that the Taub adiabat, Eq. (2.12), becomes

$$\frac{(e_2+P_2)(e_2+P_1)}{((\Gamma-1)\,e_2-P_2)^2} = \frac{(e_1+P_1)(e_1+P_2)}{((\Gamma-1)\,e_1-P_1)^2}. \tag{A.4Γ}$$

Thus we obtain a quadratic in e_2 for a specified e_1, P_1, and P_2. The ensuing equations can be written quite simply if we define

$$\tau(P_2; P_1, e_1) = \frac{(e_1+P_1)(e_1+P_2)}{((\Gamma-1)\,e_1-P_1)^2}. \tag{A.5Γ}$$

We will also require the derivative of τ with respect to P_2 which we write as

$$\tau'(P_2; P_1, e_1) = \frac{(e_1+P_1)}{((\Gamma-1)\,e_1-P_1)^2}. \tag{A.6Γ}$$

In the following equations we will omit writing out the functional dependence of τ and τ' explicitly whenever it is possible to do so without loss of ambiguity. The quadratic for e_2 can be written as

$$(1-\tau(\Gamma-1)^2)\,e_2^2 + (P_1+P_2+2\tau(\Gamma-1)\,P_2)\,e_2 + (P_1-P_2\tau)\,P_2 = 0. \tag{A.7Γ}$$

Note that

$$\tau > \frac{1}{(\Gamma-1)^2}\,\frac{(e_1+P_1)(e_1+P_2)}{e_1^2} > \frac{1}{(\Gamma-1)^2}, \tag{A.8Γ}$$

so that the coefficient of the quadratic term in Eq. (A.7Γ) always remains negative and non-zero. The worst case occurs when the pre-shock fluid is very tenuous, i.e., $e_1 \gg P_1$. Then we have $\tau \to (e_1+P_2)/(\Gamma-1)^2\,e_1$. Thus when P_2 is small compared to e_1 we have $\tau \to 1/(\Gamma-1)^2$. In practice, with the use of double-precision arithmetic this should never be a problem. With all these facts taken into account, it is easy to see that the root of Eq. (A.7Γ) that gives positive values for e_2 is given by

$$e_2 = \frac{\left(\begin{array}{l} -(P_1+P_2+2\tau(\Gamma-1)\,P_2) \\ -\sqrt{\begin{array}{l}(P_1+P_2+2\tau(\Gamma-1)\,P_2)^2 \\ \quad -4(1-\tau(\Gamma-1)^2)(P_1-P_2\tau)\,P_2\end{array}} \end{array}\right)}{2(1-\tau(\Gamma-1)^2).} \tag{A.9Γ}$$

The total derivative of e_2 with respect to P_2 can found by taking differentials of Eq. (A.7Γ) and using Eq. (A.9Γ). Thus we obtain

$$\frac{de_2}{dP_2} = -\frac{\left(\begin{array}{c}\tau'(\Gamma-1)^2\,e_2^2 - e_2 - 2(\Gamma-1)(\tau' P_2+\tau)\,e_2 \\ -P_1+2P_2\tau+P_2^2\tau'\end{array}\right)}{\left(\sqrt{\begin{array}{l}(P_1+P_2+2\tau(\Gamma-1)\,P_2)^2 \\ \quad -4(1-\tau(\Gamma-1)^2)(P_1-P_2\tau)\,P_2\end{array}}\right)}. \tag{A.10Γ}$$

APPENDIX B: SOME USEFUL EQUATIONS DERIVED FROM THE FLUID SPEED IN THE SHOCK'S REST FRAME

In this appendix we derive some useful partial and total derivatives of the fluid speeds in the shock's rest frame (Eqs. (2.14) and (2.15)). In the Riemann solver we treat the pressure at the contact discontinuity in the Riemann problem as the independent variable and the difference between the fluid velocities on either side of the contact discontinuity as the dependent variable that needs to be zeroed by an iterative root-solver. The expressions for the fluid's lab-frame velocity on each side of the contact discontinuity will involve both the pre- and post-shock fluid speeds in the rest frame of the shock propagating on that side. Thus we see that we will need expressions for the total derivative of v_1 and v_2 with respect to P_2.

A glance at Eqs. (2.14) and (2.15) shows that when $P_2 \to P_1$ we will have a problem in evaluating these expressions and any of their derivatives. Thus for stable numerical evaluation we will need one set of expressions when P_2 is significantly different from P_1 and another when $P_2 \to P_1$. In practice we have found that $|P_2 - P_1| > 0.01 \min(P_2, P_1)$ is a fair criterion to allow us to say that P_2 and P_1 are significantly different. Thus we derive the required formulae in the following two cases.

Case I. *P_2 is significantly different from P_1.* Note first that e_2 varies monotonically with P_2. Thus when P_2 is significantly different from P_1 we are assured that e_2 is significantly different from e_1. In this case we have

$$\frac{dv_1}{dP_2} = \frac{\partial v_1}{\partial P_2} + \frac{\partial v_1}{\partial e_2}\frac{de_2}{dP_2} \tag{B.1}$$

$$\frac{dv_2}{dP_2} = \frac{\partial v_2}{\partial P_2} + \frac{\partial v_2}{\partial e_2}\frac{de_2}{dP_2}, \tag{B.2}$$

where the total derivative of e_2 with respect to P_2 is given by Eq. (A.10Γ). The partial derivatives above are easy to evaluate and are given by

$$\frac{\partial v_1}{\partial P_2} = \frac{v_1}{2}\left[\frac{1}{(P_2 - P_1)} - \frac{1}{(e_1 + P_2)}\right] \tag{B.3}$$

$$\frac{\partial v_1}{\partial e_2} = -\frac{v_1}{2}\left[\frac{1}{(e_2 - e_1)} - \frac{1}{(e_2 + P_1)}\right] \tag{B.4}$$

$$\frac{\partial v_2}{\partial P_2} = \frac{v_2}{2}\left[\frac{1}{(P_2 - P_1)} + \frac{1}{(e_1 + P_2)}\right] \tag{B.5}$$

$$\frac{\partial v_2}{\partial e_2} = -\frac{v_2}{2}\left[\frac{1}{(e_2 - e_1)} + \frac{1}{(e_2 + P_1)}\right]. \tag{B.6}$$

Thus all the derivatives in Eqs. (B.1) and (B.2) have been catalogued.

Case II. *P_2 is not significantly different from P_1.* Note that when $P_2 \to P_1$ we are operating in a weak shock limit. In that limit, as shown by Landau and Lifshitz [10, Section 86], S_2 and S_1 differ only by a term that is proportional to the third order of $(P_2 - P_1)$. Synge [17] has shown that the same is true for relativistic shocks. Thus when $P_2 \to P_1$ we can assert the following (here $\Delta\rho = (\rho_2 - \rho_1)$):

$$P_2 - P_1 = \left(\frac{\Gamma P_1}{\rho_1}\right)\left(1 + \frac{(\Gamma - 1)}{2\rho_1}\Delta\rho\right)\Delta\rho \tag{B.7Γ}$$

$$e_2 - e_1 = \mu_1\left(1 + \frac{\Gamma P_1}{2\rho_1^2\mu_1}\Delta\rho\right)\Delta\rho \tag{B.8Γ}$$

$$\frac{P_2 - P_1}{e_2 - e_1} = \left(\frac{\Gamma P_1}{\rho_1\mu_1}\right) + \left(\frac{\Gamma P_1}{\rho_1\mu_1}\right)\left[\frac{(\Gamma - 1)}{2\rho_1} - \frac{\Gamma P_1}{2\rho_1^2\mu_1}\right]\Delta\rho. \tag{B.9Γ}$$

Thus whenever $P_2 \to P_1$ we can set

$$\frac{P_2 - P_1}{e_2 - e_1} = \left(\frac{\Gamma P_1}{\rho_1\mu_1}\right). \tag{B.10Γ}$$

It is also trivial to show that

$$\frac{de_2}{d\rho_2} = \mu_1, \qquad \frac{dP_2}{d\rho_2} = \frac{\Gamma P_1}{\rho_1},$$
$$\frac{de_2}{dP_2} = \frac{\rho_1\mu_1}{\Gamma P_1} \equiv \frac{1}{\alpha_1^2}, \tag{B.11Γ}$$

so that Eqs. (B.9Γ) and (B.11Γ) taken together give

$$\frac{d}{dP_2}\left(\frac{P_2 - P_1}{e_2 - e_1}\right) = \frac{1}{\mu_1}\left[\frac{(\Gamma - 1)}{2\rho_1} - \frac{\Gamma P_1}{2\rho_1^2\mu_1}\right]. \tag{B.12Γ}$$

Thus in the limit where $P_2 \to P_1$ we have

$$\frac{dv_1}{dP_2} = \frac{(\Gamma + 1)}{4\alpha_1\rho_1\mu_1} - \frac{3\alpha_1}{4\rho_1\mu_1} \tag{B.13Γ}$$

and

$$\frac{dv_2}{dP_2} = \frac{(\Gamma - 3)}{4\alpha_1\rho_1\mu_1} + \frac{\alpha_1}{4\rho_1\mu_1}. \tag{B.14Γ}$$

If the shocks are both weak and non-relativistic, i.e., $P_2 \to P_1$, $\alpha_1 \ll 1$, and $\mu_1 \to 1$, we obtain

$$\frac{d(v_2 - v_1)}{dP_2} = -\frac{1}{\alpha_1\rho_1}. \tag{B.15Γ}$$

Note that the denominator on the right-hand side of

(B.15Γ) is just the lagrangian sound speed. This is exactly what van Leer [19] obtains in the non-relativistic case when the shock is a weak shock. This shows that in the non-relativistic limit when considering weak shocks our equations reduce trivially to those of van Leer [19]. In Appendix C we show that in the non-relativistic limit this is true for shocks of arbitrary strength.

APPENDIX C: EQUIVALENCE TO THE LAGRANGIAN FORMULATION OF VAN LEER [19]

Our method of deriving the Riemann solver iteration is very different from the lagrangian method devised by van Leer [19]. Thus we show that in the non-relativistic limit our method reduces to that of van Leer [19] (with simple waves restricted to being shocks, Colella [2]).

In the non-relativistic limit, using arguments outlined at the beginning of Section IV we obtain the post-shock x-velocities in the lab-frame as

$$v_{x2l} = v_{x1l} - (v_{1L} - v_{2L}) \tag{C.1}$$

$$v_{x2r} = v_{x1r} + (v_{1R} - v_{2R}). \tag{C.2}$$

It may also be noted that Eqs. (3.11) and (3.13) reduce to Eq. (C.1) in the non-relativistic limit. Similarly, Eqs. (3.23) and (3.25) reduce to Eq. (C.2). The derivatives of Eqs. (C.1) and (C.2) above, with respect to the post-shock pressure P_2, are given by

$$\frac{dv_{x2l}}{dP_2} = -\frac{d(v_{1L} - v_{2L})}{dP_2} \tag{C.3}$$

$$\frac{dv_{x2r}}{dP_2} = \frac{d(v_{1R} - v_{2R})}{dP_2}. \tag{C.4}$$

In the non-relativistic limit the equation relating the rest-frame velocity jump to the post-shock pressure is given by Eq. (89.5) of Landau and Lifshitz [10],

$$v_1 - v_2 = \sqrt{2/\rho_1}\,\frac{(P_2 - P_1)}{\sqrt{(\Gamma - 1)\,P_1 + (\Gamma + 1)\,P_2}}, \tag{C.5Γ}$$

and its total derivative with respect to pressure P_2 is given by

$$\frac{d(v_1 - v_2)}{dP_2} = \frac{1}{\sqrt{2\rho_1}}\,\frac{(3\Gamma - 1)\,P_1 + (\Gamma + 1)\,P_2}{[(\Gamma - 1)\,P_1 + (\Gamma + 1)\,P_2]^{3/2}}. \tag{C.6Γ}$$

In the Eqs. (C.5Γ) and (C.6Γ), above, P_1 has to be suitably set to be the pressure on the left or the right of the Riemann problem. This allows us to evaluate the right-hand sides of Eqs. (C.3) and (C.4). When Eqs. (C.1), (C.2), (C.3), and (C.4) are put into Eq. (4.2) we obtain the required non-relativistic iteration scheme. Our formulation is more like an eulerian formulation, in the sense that it is done in one single lab-frame with the help of simple Galilean transforms. It is interesting to note that it yields a non-relativistic iteration scheme in just a few steps.

In the lagrangian formulation of van Leer [19] W, P^*, and u^* denote the lagrangian shock speed of the specified shock, the pressure behind the shock, and the lagrangian velocity behind the shock, respectively. The difference in post-shock velocities is the same whether formulated in a lagrangian or lab-frame. Thus the square-bracketed term in the numerator on the right-hand side of Eq. (4.2) remains the same. We will show that so does the denominator. Equation (31Γ) of van Leer [19] gives (in our notation)

$$W = \sqrt{\Gamma P_1 \rho_1}\,[1 + ((\Gamma + 1)/(2\Gamma))(P_2 - P_1)/P_1]^{1/2} \tag{C.7Γ}$$

which can also be written as

$$W = \sqrt{\rho_1/2}\,[(\Gamma - 1)\,P_1 + (\Gamma + 1)\,P_2]^{1/2}. \tag{C.8Γ}$$

The absolute value of the slope of the Hugoniot is denoted by Z and is given by Eq. (A.2) of van Leer [19] (where we consider only the shock wave not the rarefaction wave):

$$Z \equiv \left|\frac{du^*}{dP^*}\right| = \frac{W^2 + \Gamma P_1 \rho_1}{2W^3}. \tag{C.9Γ}$$

Substituting Eq. (C.8Γ) into (C.9Γ) we see that it reduces exactly to the right-hand side of Eq. (C.6Γ). Thus the slopes of the Hugoniot in the two formulations match up, as they should. The iteration scheme of van Leer [19] is given by his Eq. (A.1) which is written as

$$P^{*(n+1)} = P^{*(n)} - Z_-^{(n)} Z_+^{(n)} (u_+^{*(n)} - u_-^{*(n)})/(Z_-^{(n)} + Z_+^{(n)}), \tag{C.10}$$

where the subscripts $-$ and $+$ denote values for the left- and right-going shocks, respectively. Thus after making note of the negative sign in Eq. (C.3) it is trivial to see that the two formulations yield exactly the same iteration scheme. Thus the equivalence is established.

REFERENCES

1. D. Balsara, in preparation.
2. P. Colella, *SIAM J. Sci. Stat. Comput.* **3**, 76 (1982).
3. P. Colella and P. R. Woodward, *J. Comput. Phys.* **54**, 174 (1984).
4. P. Colella and H. Glaz, *J. Comput. Phys.* **59**, 264 (1985).
5. A. Harten, *J. Comput. Phys.* **49**, 357 (1983).

6. A. Harten and J. M. Hyman, *J. Comput. Phys.* **50**, 235 (1983).
7. A. Harten, S. Osher, B. Engquist, and S. Chakravarthy, *J. Comput. Phys.* **71**, 231 (1987).
8. J. F. Hawley, L. L. Smarr, and J. R. Wilson, *Astrophys. J.* **277**, 296 (1984).
9. J. F. Hawley, L. L. Smarr, and J. R. Wilson, *Astrophys. J. Suppl.* **55**, 211 (1984).
10. L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon, Elmsford, NY, 1987).
11. D. Mihalas and B. W. Mihalas, *Foundations of Radiation Hydrodynamics* (Oxford Univ. Press, Oxford, 1984).
12. M. L. Norman and K.-H. Winkler, *Astrophysical Radiation Hydrodynamics* (Reidel, Dordrecht, 1986).
13. S. Osher and S. Chakravarthy, *SIAM J. Numer. Anal.* **21**, 955 (1984).
14. S. Osher and F. Solomon, *Math. Comput.* **38**, 339 (1982).
15. [illegible] *Numerical Recipes* (Cambridge Univ. Press, Cambridge, UK, 1986).
16. P. L. Roe, *J. Comput. Phys.* **43**, 357 (1981).
17. J. L. Synge, *The Relativistic Gas* (North Holland, Amsterdam, 1957).
18. A. H. Taub, *Phys. Rev.* **74**, 3 (1948).
19. B. van Leer, *J. Comput. Phys.* **32**, 101 (1979).
20. P. R. Woodward and P. Colella, *J. Comput. Phys.* **54**, 115 (1984).